Hydrostatic equilibrium and stellar structure in f(R)-gravity

Salvatore Capozziello

Università di Napoli "Federico II"
and
INFN Sez. di Napoli

In collaboration with

M. De Laurentis, S.D. Odintsov, A. Stabile

PRD 83, 064004 (2011)
PRD 85, 044022 (2012)

Outlines

- Hydrostatic equilibrium of stellar structures
- The Newtonian limit of f(R)-gravity
- Stellar hydrostatic equilibrium in f(R)-gravity
- Solution of the standard and modified Lanè-Emden equations
- Discussion and conclusions
- Next steps

Setting the problem

- Several open questions in modern Astrophysics ask for new paradigms.
- No evidence of Dark Energy and Dark Matter at fundamental level (LHC, astroparticle physics, ground based experiments…).
- Such problems could be framed extending GR at infrared scales
- GR does not work at ultraviolet scales (no quantum gravity theory up to now).
- f(R)-gravity as minimal extension but other modifications are possible.
- Several stellar structures cannot be addressed by the standard theory of stellar evolution (magnetars, variable stars, etc..)
- Big issue: Is it possible to revise stellar theory in view of extended gravity?

Hydrostatic equilibrium of stellar structures

The condition of hydrostatic equilibrium in Newtonian dynamics is

$$\frac{dp}{dr} = \frac{d\Phi}{dr}\rho$$

- p is the pressure,
- Φ is the gravitational potential,
- ρ is the density

The Poisson equation $$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\Phi}{dr}\right) = -4\pi G\rho$$

Since we are taking into account only static and stationary situations, here we consider only time independent solutions

In general, the temperature τ appears and the density ρ satisfies an equation of state of the form

$$\rho = \rho(p,\tau)$$

R. Kippenhahn and A. Weigert, Stellar Structures and Evolution (Springer-Verlag, Berlin, 1990).

Hydrostatic equilibrium of stellar structures

We assume that there exists a polytropic relation between p and ρ of the form

$$p = K\rho^{\gamma}$$

- K is the polytropic constant that can be obtained as a combination of fundamental constants
- The constant γ is the polytropic exponent.
- Note that $\Phi > 0$ is in the interior of the model, since we define the gravitational potential as $-\Phi$

Inserting the polytropic equation of state, we obtain

$$\frac{d\Phi}{dr} = \gamma K \rho^{\gamma-2} \frac{d\rho}{dr}$$

Hydrostatic equilibrium of stellar structures

For $\gamma \neq 1$, the above equation can be integrated giving

$$\frac{\gamma K}{\gamma - 1}\rho^{\gamma-1} = \Phi \rightarrow \rho = \left[\frac{\gamma - 1}{\gamma K}\right]^{1/(\gamma-1)} \Phi^{1/(\gamma-1)} \doteq A_n \Phi^n$$

- We have chosen the integration constant to give $\Phi = 0$ at surface ($\rho = 0$)
- $n = \frac{1}{\gamma - 1}$ Is the polytropic index

Inserting the above relation into the Poisson equation, we obtain a differential equation for the gravitational potential

$$\frac{d^2\Phi}{dr^2} + \frac{2}{r}\frac{d\Phi}{dr} = -4\pi G A_n \Phi^n$$

Hydrostatic equilibrium of stellar structures

Let us define now the dimensionless variables:

$$z = |\mathbf{x}|\sqrt{\frac{\chi A_n \Phi_c^{n-1}}{2}} \qquad w(z) = \frac{\Phi}{\Phi_c} = \left(\frac{\rho}{\rho_c}\right)^{1/n}$$

- Where the subscript c refers to the center of the star and the relation between ρ and Φ
- At the center ($r = 0$), we have $z = 0$, $\Phi = \Phi_c$, , $\rho = \rho_c$ and therefore $w = 1$

Then we obtain the standard Lané-Emden equation describing the hydrostatic equilibrium of stellar structures in the Newtonian theory

$$\boxed{\frac{d^2w}{dz^2} + \frac{2}{z}\frac{dw}{dz} + w^n = 0.}$$

The Newtonian limit of f(R) - gravity

Let us start with a general class of Extended Theories of Gravity (ETG) given by the action

$$\mathcal{A} = \int d^4x\sqrt{-g}[f(R) + \mathcal{X}\mathcal{L}_m],$$

Varying the action with respect to the metric we obtain the field equations

$$f'R_{\mu\nu} - \frac{f}{2}g_{\mu\nu} - f'_{;\mu\nu} + g_{\mu\nu}\Box f' = \mathcal{X}T_{\mu\nu}$$

$$3\Box f' + f'R - 2f = \mathcal{X}T,$$

S. Capozziello, M. De Laurentis Phys. Rep. 509, 167-321 (2011)
S. Capozziello , M. Francaviglia, Gen. Relativ. Gravit. 40, 357 (2007)

The Newtonian limit of f(R) - gravity

In order to achieve the Newtonian limit of the theory the metric tensor has to be approximated as follows:

$$g_{\mu\nu} \sim \begin{pmatrix} 1 - 2\Phi(t, \mathbf{x}) + \mathcal{O}(4) & \mathcal{O}(3) \\ \mathcal{O}(3) & -\delta_{ij} + \mathcal{O}(2) \end{pmatrix},$$

The Ricci scalar formally becomes

$$R \sim R^{(2)}(t, \mathbf{x}) + \mathcal{O}(4).$$

The **n**-th derivative of Ricci function can be developed as

$$f^n(R) \sim f^n(R^{(2)} + \mathcal{O}(4)) \sim f^n(0) + f^{n+1}(0)R^{(2)} + \mathcal{O}(4)$$

here $\mathbf{R}^{n}$ denotes a quantity of order **O(n)**

S. Capozziello, A. Stabile, and A. Troisi, Phys. Rev. D 76, 104019 (2007)

The Newtonian limit of f(R) - gravity

Field equations at O (2)*-order, that is at the Newtonian level, are*

$$R^{(2)}_{tt} - \frac{R^{(2)}}{2} - f''(0) \triangle R^{(2)} = \mathcal{X} T^{(0)}_{tt}$$

$$-3 f''(0) \triangle R^{(2)} - R^{(2)} = \mathcal{X} T^{(0)},$$

- Δ *is the Laplacian in the flat space* $R_{tt} = \Delta \Phi$ *and, for the sake of simplicity, we set* f' (R)_0 = 1

We recall that the energy-momentum tensor for a perfect fluid is

$$T_{\mu\nu} = (\epsilon + p) u_\mu u_\nu - p g_{\mu\nu}$$

- *p is the pressure and is the energy density*

The Newtonian limit of $f(R)$ - gravity

Being the pressure contribution negligible in the field equations in the Newtonian approximation, we have

modified Poisson equation

$$\begin{cases} \triangle\,\Phi + \dfrac{R^{(2)}}{2} + f''(0)\,\triangle\,R^{(2)} = -\mathcal{X}\rho \\ 3f''(0)\,\triangle\,R^{(2)} + R^{(2)} = -\mathcal{X}\rho, \end{cases}$$

- ρ is now the mass density

For $f''(R) = 0$ we have the standard Poisson equation

$$\triangle\Phi = -4\pi G\rho$$

This means that as soon as the second derivative of $f(R)$ is different from zero, deviations from the Newtonian limit of GR emerge

Stellar hydrostatic equilibrium in f(R) - gravity

From the Bianchi identities we have

$$T^{\mu\nu}{}_{;\mu} = 0 \rightarrow \frac{\partial p}{\partial x^k} = -\frac{1}{2}(p+\epsilon)\frac{\partial \ln g_{tt}}{\partial x^k}.$$

- If the dependence on the temperature is negligible, this relation can be introduced into field equations, which becomes a system of three equations for p, Φ and R.(2) and can be solved without the other structure equations.

Let us suppose that matter still satisfies a polytropic equation

$$p = K\rho^{\gamma}$$

Stellar hydrostatic equilibrium in f(R) - gravity

We obtain an integral-differential equation for the gravitational potential, that is

$$\triangle\,\Phi(\mathbf{x}) + \frac{2\mathcal{X}A_n}{3}\Phi(\mathbf{x})^n$$

$$= -\frac{m^2\mathcal{X}A_n}{6}\int d^3\mathbf{x}'\,\mathcal{G}(\mathbf{x},\mathbf{x}')\Phi(\mathbf{x}')^n$$

- ★ $\mathcal{G}(\mathbf{x},\mathbf{x}') = -\frac{1}{4\pi}\frac{e^{-m|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}$ *is the Green function*
- ★ $m^2 = -\frac{1}{3f''(0)}$

Stellar hydrostatic equilibrium in f(R) - gravity

Adopting again the dimensionless variables

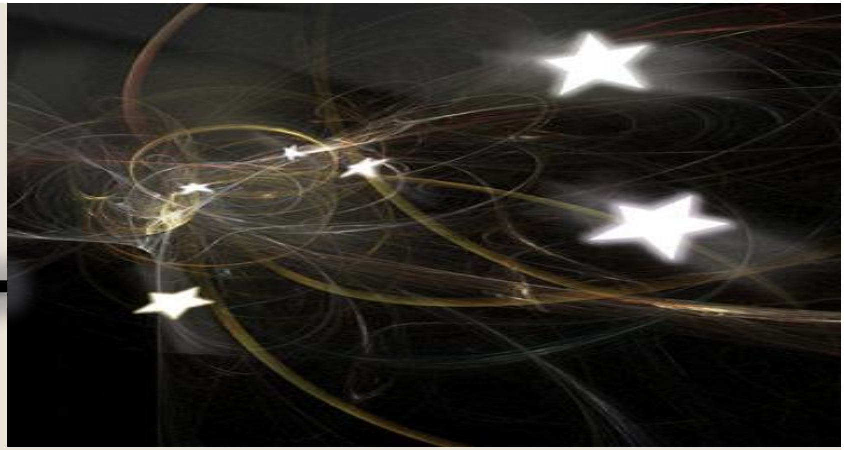
$$z = \frac{|\mathbf{x}|}{\xi_0} \qquad w(z) = \frac{\Phi}{\Phi_c}$$

$$\xi_0 \doteq \sqrt{\frac{3}{2\mathcal{X}A_n\Phi_c^{n-1}}}$$ *is a characteristic length linked to stellar radius* ξ

The Lanè-Emden in f(R)-gravity becomes

$$\frac{d^2w(z)}{dz^2} + \frac{2}{z}\frac{dw(z)}{dz} + w(z)^n = \frac{m\xi_0}{8}\frac{1}{z}\int_0^{\xi/\xi_0} dz' z' \left\{ e^{-m\xi_0|z-z'|} - e^{-m\xi_0|z+z'|} \right\} w(z')^n$$

Solutions of the standard and modified Lanè-Emden equations



The task is now to solve the modified Lane´-Emden equation and compare its solutions to those of the standard Newtonian theory

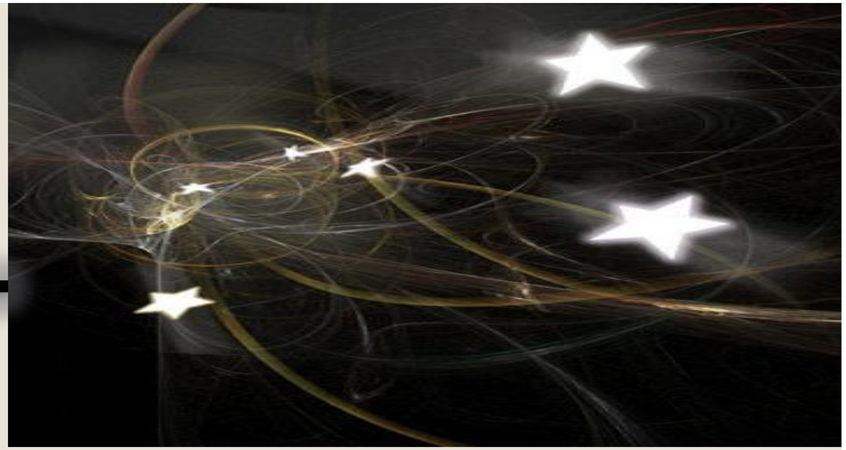
Only for three values of **n**, the classical solutions have analytical expression

$$n = 0 \rightarrow w_{\mathrm{GR}}^{(0)}(z) = 1 - \frac{z^2}{6}$$

$$n = 1 \rightarrow w_{\mathrm{GR}}^{(1)}(z) = \frac{\sin z}{z}$$

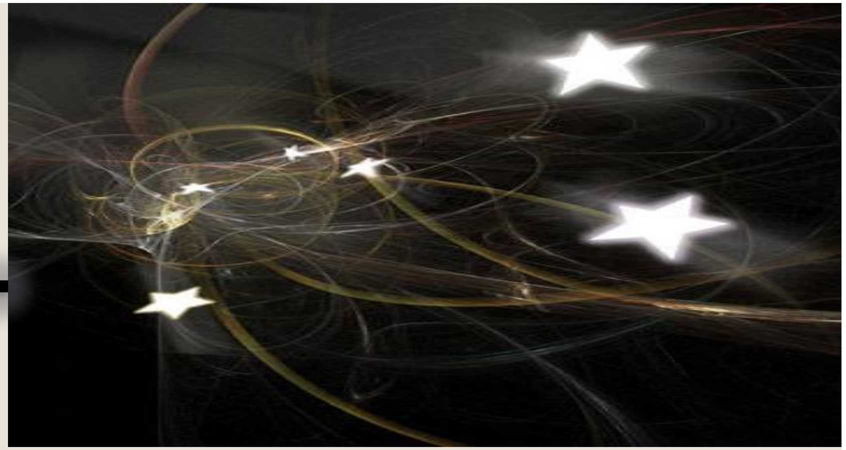
$$n = 5 \rightarrow w_{\mathrm{GR}}^{(5)}(z) = \frac{1}{\sqrt{1 + \frac{z^2}{3}}}.$$

Solutions of the standard and modified Lanè-Emden equations



- The surface of the polytrope of index n is defined by the value $z = z^{(n)}$. where $\rho = 0$ and thus $w = 0$
- For $n = 0$ and $n = 1$ the surface is reached for a finite value of $z = z^{(n)}$
- The case $n = 5$ yields a model of infinite radius
- It can be shown that for $n<5$ the radius of polytropic models is finite; for $n>5$ they have infinite radius
- One finds $z^{(0)}{}_{GR} = \sqrt{6}$, $z^{(1)}{}_{GR} = \pi$, $z^{(5)}{}_{GR} = \infty$
- A general property of the solutions is that $z^{(n)}$ grows monotonically with the polytropic index **n**

Solutions of the standard and modified Lanè-Emden equations



Apart from the three cases where analytic solutions are known, the classical Lane´-Emden has to be solved numerically, considered with the expression for the neighborhood of the center

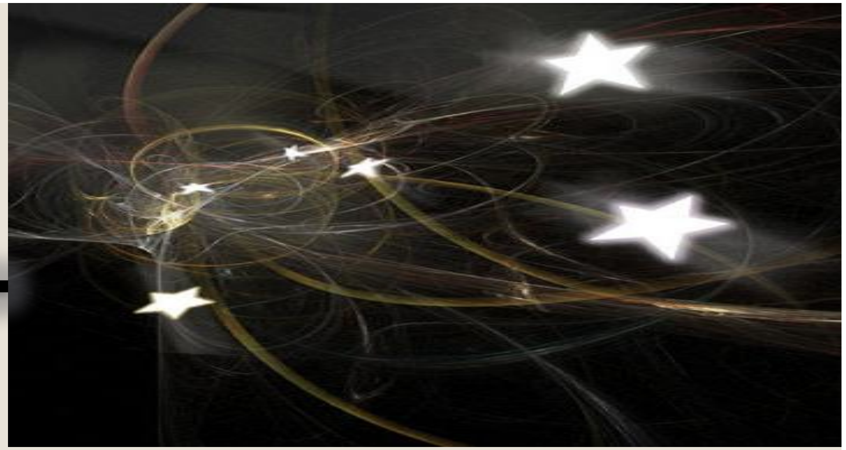
$$w_{\mathrm{GR}}^{(n)}(z) = \sum_{i=0}^{\infty} a_i^{(n)} z^i$$

at lowest orders, a classification of solutions by the index **n**, that is

$$w_{\mathrm{GR}}^{(n)}(z) = 1 - \frac{z^2}{6} + \frac{n}{120} z^4 + \ldots$$

- The case $\gamma=5/3$ and $n= 3/2$ is the nonrelativistic limit while the case $\gamma=4/3$ and $n = 3$ is the relativistic limit of a completely degenerate gas.

Solutions of the standard and modified Lanè-Emden equations



For the modified Lane´-Emden, we have an exact solution for n = 0, in fact

$$w^{(0)}_{f(R)}(z) = 1 - \frac{z^2}{8} + \frac{(1+m\xi)e^{-m\xi}}{4m^2\xi_0^2}\left[1 - \frac{\sinh m\xi_0 z}{m\xi_0 z}\right],$$

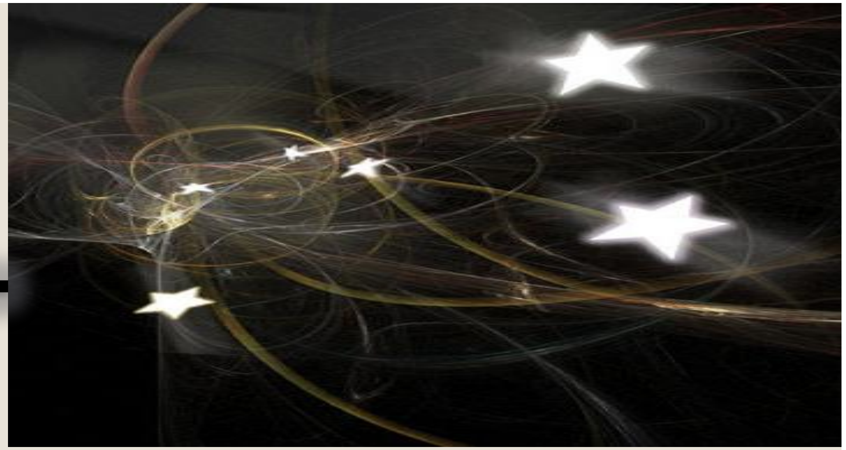
Where the boundary conditions **w(0) = 1** and **w'(0) = 0** are satisfied

- A comment on the GR limit (that is **f(R) → R**) of above solution is necessary.

In fact, when we perform the limit m → ∞ we do not recover exactly $w^{(0)}_{GR}(z)$.
The difference is in the definition of quantity ξ_0

In GR it is $\xi_0 = \sqrt{\frac{2}{\chi A_n \Phi_c^{n-1}}}$

Solutions of the standard and modified Lanè-Emden equations



The point $z^{(o)}_{f(R)}$ is calculated by imposing $W^{(o)}_{f(R)}(z^{(o)}_{f(R)})=0$ and by considering the Taylor expansion

$$\frac{\sinh m\xi_0 z}{m\xi_0 z} \sim 1 + \frac{1}{6}(m\xi_0 z)^2 + \mathcal{O}(m\xi_0 z)^4$$

We obtain $z^{(0)}_{f(R)} = \frac{2\sqrt{6}}{\sqrt{3+(1+m\xi)e^{-m\xi}}}$

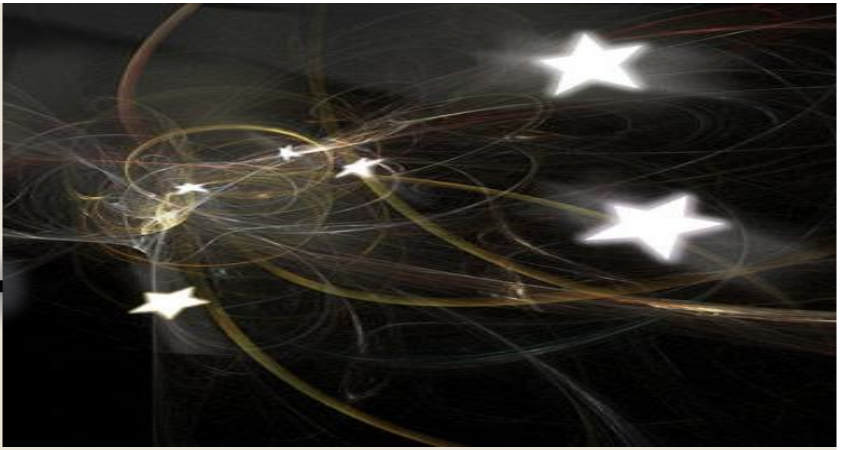
Since the stellar radius ξ is given by definition $\xi = \xi_o \; z^{(o)}_{f(R)}$ we obtain

$$\xi = \sqrt{\frac{3\Phi_c}{2\pi G}}\frac{1}{\sqrt{1+\frac{1+m\xi}{3}e^{-m\xi}}}.$$

By solving numerically the constraint, we find the modified expression of the radius

If $\mathbf{m} \rightarrow \infty$ we have the standard expression valid for the Newtonian limit of GR

Solutions of the standard and modified Lanè-Emden equations



In the f(R)-gravity case, for n=0, the radius is smaller than in GR

In the case n= 1 we obtain

$$\frac{d^2\tilde{w}(z)}{dz^2} + \tilde{w}(z) = \frac{m\xi_0}{8}\int_0^{\xi/\xi_0} dz' \times \left\{e^{-m\xi_0|z-z'|} - e^{-m\xi_0|z+z'|}\right\}\tilde{w}(z'),$$

$$\tilde{w} = zw$$

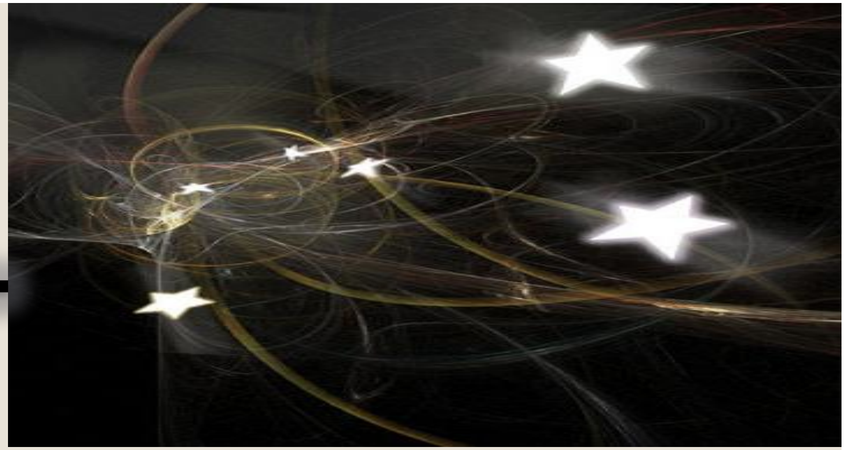
If we perturb this equations we have $\tilde{w}^{(1)}_{f(R)}(z) \sim \tilde{w}^{(1)}_{\mathrm{GR}}(z) + e^{-m\xi}\Delta\tilde{w}^{(1)}_{f(R)}(z).$

The coefficient $e^{-m\xi} < 1$ is the parameter with respect to which we perturb

And then

$$\frac{d^2\Delta\tilde{w}^{(1)}_{f(R)}(z)}{dz^2} + \Delta\tilde{w}^{(1)}_{f(R)}(z) = \frac{m\xi_0 e^{m\xi}}{8}\int_0^{\xi/\xi_0} dz'\left\{e^{-m\xi_0|z-z'|} - e^{-m\xi_0|z+z'|}\right\}\tilde{w}^{(1)}_{\mathrm{GR}}(z')$$

Solutions of the standard and modified Lanè-Emden equations



And the solutions is easily found to be

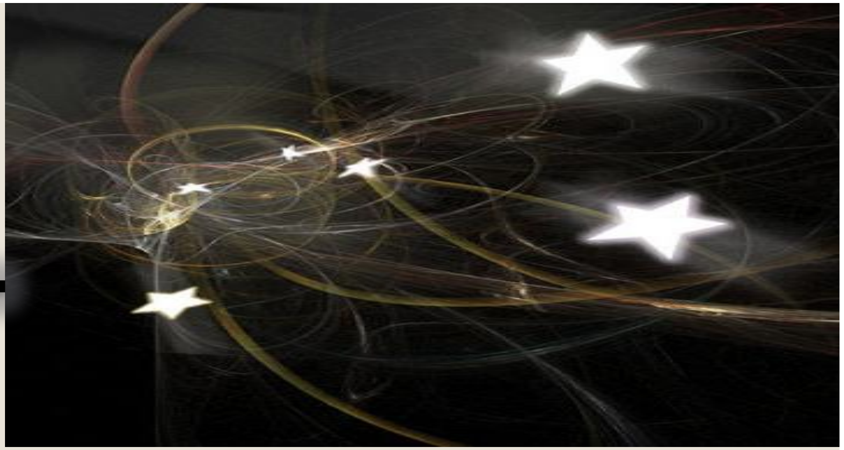
$$w^{(1)}_{f(R)}(z) \sim \frac{\sin z}{z}\left\{1+\frac{m^2\xi_0^2}{8(1+m^2\xi_0^2)}\left[1+\frac{2e^{-m\xi}}{1+m^2\xi_0^2}\right.\right.$$
$$\left.\left.\times\left(\cos\xi/\xi_0+m\xi_0\sin\xi/\xi_0\right)\right]\right\}$$
$$-\frac{m^2\xi_0^2}{8(1+m^2\xi_0^2)}\left[\frac{2e^{-m\xi}}{1+m^2\xi_0^2}\right.$$
$$\left.\times\left(\cos\xi/\xi_0+m\xi_0\sin\xi/\xi_0\right)\frac{\sinh m\xi_0 z}{m\xi_0 z}+\cos z\right].$$

Also in this case, for **m→∞**, we do not recover exactly $w^{(1)}{}_{GR}(z)$

The reason is the same of the previous n = 0 case

Analytical solutions for other values of n are not available

Solutions of the standard and modified Lanè-Emden equations



Finally we report the gravitational potential profile generated by a spherically symmetric source of uniform mass with radius ξ

We can impose a mass density of the form $\rho = \frac{3M}{4\pi\xi^3}\Theta(\xi - |\mathbf{x}|)$,

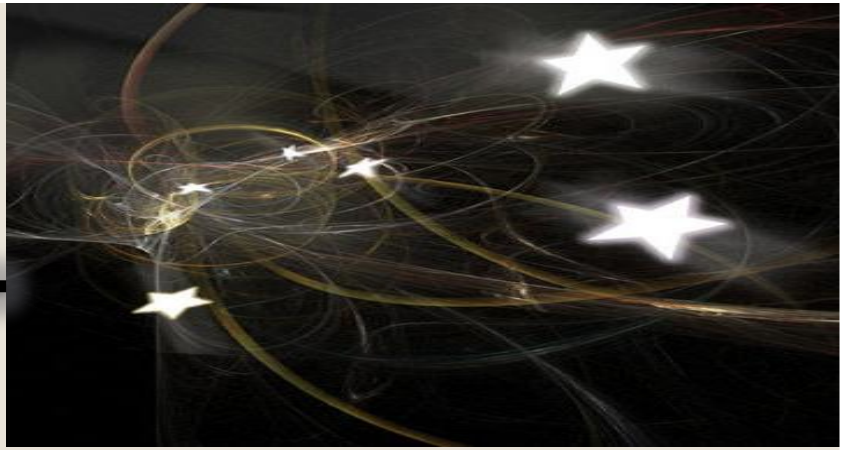
- Θ is the Heaviside function and M is the mass

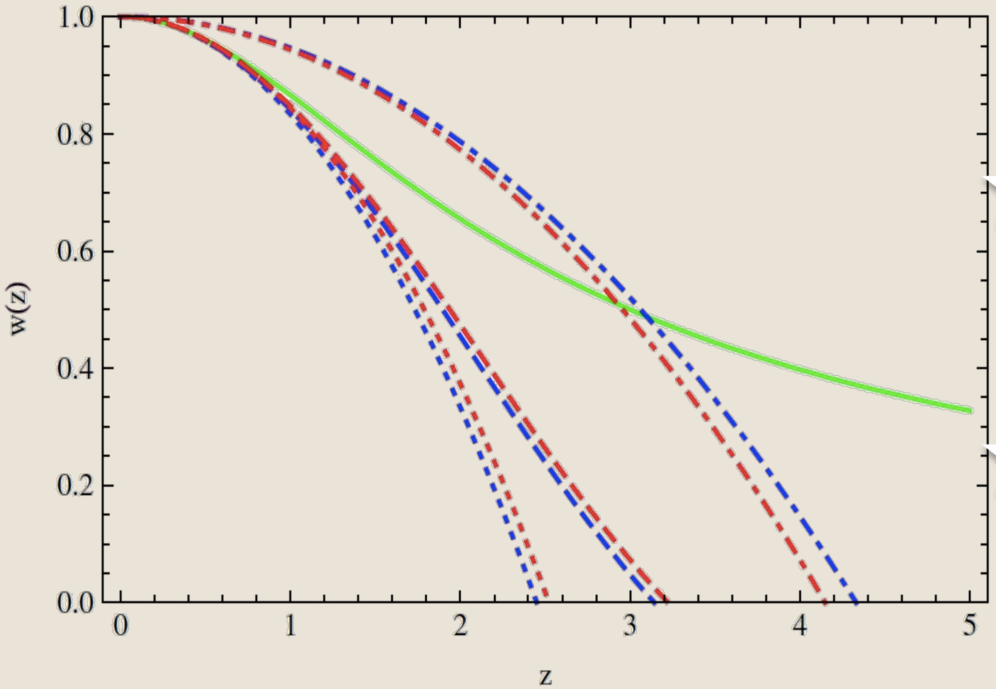
By solving field equations inside the star and considering the boundary conditions **w.(0) = 1** and w'(0)=0, we get

$$w_{f(R)}(z) = \left[\frac{3}{2\xi} + \frac{1}{m^2\xi^3} - \frac{e^{-m\xi}(1+m\xi)}{m^2\xi^3}\right]^{-1}\left[\frac{3}{2\xi} + \frac{1}{m^2\xi^3} - \frac{\xi_0^2 z^2}{2\xi^3} - \frac{e^{-m\xi}(1+m\xi)}{m^2\xi^3}\frac{\sinh m\xi_0 z}{m\xi_0 z}\right].$$

In the limit **m→∞** we recover the GR case $w_{\mathrm{GR}}(z) = 1 - \frac{\xi_0^2 z^2}{3\xi^2}$

Solutions of the standard and modified Lanè-Emden equations





- Plot of solutions (blue lines) of standard Lane´-Emden: $w^{(0)}{}_{GR}(z)$ (dotted line) and $w^{(1)}{}_{GR}(z)$ (dashed line). The green line corresponds to $w^{(5)}{}_{GR}(z)$
- The red lines are the solutions of modified Lane´-Emden: $w^{(0)}{}_{f(R)}(z)$ (dotted line) and $w^{(1)}{}_{f(R)}(z)$ (dashed line).
- The blue dashed-dotted line is the potential derived from GR $w_{GR}(z)$ and the red dashed dotted line is the potential derived from f(R) gravity for a uniform spherically symmetric mass distribution
- From a rapid inspection of these plots, the differences between GR and **f(R)** gravitational potentials are clear and the tendency is that at larger radius **z** they become more evident.

Discussion and Conclusions

- The hydrostatic equilibrium of a stellar structure in the framework of $f(R)$ gravity has been considered.
- Adopting a polytropic equation of state relating the mass density to the pressure, we derive the modified Lane´-Emden equation and its solutions for $n = 0,1$ which can be compared to the analogous solutions coming from the Newtonian limit of GR
- When we consider the limit $f(R) \rightarrow R$, we obtain the standard hydrostatic equilibrium theory coming from GR
- A peculiarity of $f(R)$ gravity is the nonviability of the Gauss theorem, and then the modified Lane´-Emden equation is an integro-differential equation Where the mass distribution plays a crucial role
- The correlation between two points in the star is given by a Yukawa-like term of the corresponding Green function

Discussion and Conclusions

- These solutions have been matched with those coming from GR and the corresponding density radial profiles have been derived
- In the case $n = 0$, we find an exact solution, while, for $n = 1$, we used a perturbative analysis with respect to the solution coming from GR
- It is possible to demonstrate that density radial profiles coming from $f(R)$ gravity analytic models and close to those coming from GR are compatible
- This result rules out some wrong claims in the literature stating that $f(R)$ gravity is not compatible with self-gravitating systems

Next Steps

- The next step is to derive self-consistent numerical solutions of the modified Lane´-Emden equation and build up realistic star models where further values of the polytropic index n and other physical parameters, e.g. temperature, opacity, transport of energy, are considered.

- These models are a challenging task, since, up to now, there is no self-consistent, final explanation for compact objects (e.g. neutron stars) with masses larger than Volkoff mass, while observational evidence widely indicates these objects. Another important issue is that such an approach could allow to address dynamics of systems like Wolf-Rayet stars, magnetars and oscillating stars.

Work in progress...see Mariafelicia Talk!